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On Canonical-Type Connections on Almost Contact Complex Riemannian Manifolds

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Dedicated to my academic teacher Prof. Dimitar Mekerov on the occasion of his 70th birthday

Abstract. We consider a pair of smooth manifolds, which are the counterparts in the even-dimensional and odd-dimensional cases. They are separately an almost complex manifold with Norden metric and an almost contact manifolds with B-metric, respectively. They can be combined as the so-called almost contact complex Riemannian manifold. This paper is a survey with additions of results on differential geometry of canonical-type connections (i.e. metric connections with torsion satisfying a certain algebraic identity) on the considered manifolds.

1. Introduction

The geometry of almost Hermitian manifolds (N, J, h) is well developed. As it is known, P. Gauduchon gives in [10] a unified presentation of a so-called *canonical* class of (almost) Hermitian connections, considered by P. Libermann in [14]. Let us recall, a linear connection D is called *Hermitian* if it preserves the Hermitian metric h and the almost complex structure J, $Dh = DJ = 0$. The *potential* of D (with respect to the Levi-Civita connection ∇), denoted by Q, is defined by the difference $D - \nabla$. The connection D preserves the metric and therefore is completely determined by its *torsion* T. According to [3, 33, 35], the two spaces of all torsions and of all potentials are isomorphic as $O(n)$ representations and an equivariant bijection is the following

$$T(x, y, z) = Q(x, y, z) - Q(y, x, z), \tag{1}$$

$$2Q(x, y, z) = T(x, y, z) - T(y, z, x) + T(z, x, y). \tag{2}$$

Following E. Cartan [3], there are studied the algebraic types of the torsion tensor for a metric connection, i.e. a linear connection preserving the metric.

On an almost Hermitian manifold, a Hermitian connection is called *canonical* if its torsion T satisfies the following conditions: [10]

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1) the component of T satisfying the Bianchi identity and having the property $T(J\cdot, J\cdot) = T(\cdot,\cdot)$ vanishes;

2) for some real number t, it is valid $(\mathfrak{S}T)^+ = (1-2t)(\mathrm{d}\Omega)^+(J\cdot, J\cdot, J\cdot)$, where $\mathfrak{S}$ denotes the cyclic sum by three arguments and $(\mathrm{d}\Omega)^+$ is the part of type $(2,1)+(1,2)$ of the differential $\mathrm{d}\Omega$ for the Kähler form $\Omega = g(J\cdot,\cdot)$.

According to [10], there exists an one-parameter family $\{\nabla^t\}_{t\in\mathbb{R}}$ of canonical Hermitian connections $\nabla^t = t\nabla^1 + (1-t)\nabla^0$, where ∇^0 and ∇^1 are the *Lichnerowicz first and second canonical connections* [15], respectively.

An object of our interest is the class of manifolds with Norden-type metrics.

In comparison, the action of the almost complex structure with respect to the Hermitian metric (respectively, the Norden metric) on the tangent spaces of the almost complex manifold is an isometry (respectively, an anti-isometry). The latter manifolds are known as *generalized B-manifolds* [11] or *almost complex manifolds with Norden metric* [4] or *complex Riemannian manifolds* [13]. The Norden metric is a pseudo-Riemannian metric of neutral signature whereas the Hermitian metric is Riemannian.

In the odd-dimensional case, the additional direction is spanned by a vector field ξ. Then its dual 1-form η determines a codimension one distribution $H = \ker(\eta)$ endowed with an almost complex structure φ. Then we have an almost contact structure (φ, ξ, η). If the almost complex structure is equipped with a Hermitian metric then the almost contact manifold is called *metric*. In the case when the restriction of the metric on H is a Norden metric then we deal with an *almost contact manifold with B-metric* (or an *almost contact complex Riemannian manifold*). Any B-metric as an odd-dimensional counterpart of a Norden metric is a pseudo-Riemannian metric of signature $(n+1, n)$.

The goal of the present paper is to survey the research on canonical-type connections in the case of Norden-type metrics as well as some additions and generalizations are made. In Section 2 we consider the even-dimensional case and in Section 3 — the odd-dimensional one.

Notation 1.1. *(a) The notation* $\underset{x,y,z}{\mathfrak{S}}$ *(or simply* $\mathfrak{S}$*) means the cyclic sum by the three arguments* x, y, z*; e.g.,* $\underset{x,y,z}{\mathfrak{S}} F(x,y,z) = F(x,y,z) + F(y,z,x) + F(z,x,y)$*;*

(b) For the sake of brevity, we shall use the notation $\{A(x,y,z)\}_{[x\leftrightarrow y]}$ *for the difference* $A(x,y,z) - A(y,x,z)$ *and* $\{A(x,y,z)\}_{(x\leftrightarrow y)}$ *for the sum* $A(x,y,z) + A(y,x,z)$*, where* A *is an arbitrary tensor of type* $(0,3)$*;*

(c) We shall use double subscripts separated by the symbol /. The former and latter subscripts regarding this symbol correspond to the upper and down signs plus and minus (or, = and ≠) in the same equality, respectively. For example, the notation $\mathcal{F}_{8/9} : F(x,y,z) = F(x,y,\xi)\eta(z) + F(x,z,\xi)\eta(y),\ F(x,y,\xi) = \pm F(y,x,\xi) = F(\varphi x,\varphi y,\xi)$ *means* $\mathcal{F}_8 : F(x,y,z) = F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\ F(x,y,\xi) = F(y,x,\xi) = F(\varphi x,\varphi y,\xi)$ *and* $\mathcal{F}_9 : F(x,y,z) = F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\ F(x,y,\xi) = -F(y,x,\xi) = F(\varphi x,\varphi y,\xi)$*. Similarly,* $\mathcal{T}_{1/2} : T(\xi,y,z) = T(x,y,\xi) = 0,\ T(x,y,z) = -T(\varphi x,\varphi y,z) = -T(x,\varphi y,\varphi z),\ t \overset{\neq}{=} 0$ *means* $\mathcal{T}_1 : T(\xi,y,z) = T(x,y,\xi) = 0,\ T(x,y,z) = -T(\varphi x,\varphi y,z) = -T(x,\varphi y,\varphi z),\ t \neq 0$ *and* $\mathcal{T}_2 : T(\xi,y,z) = T(x,y,\xi) = 0,\ T(x,y,z) = -T(\varphi x,\varphi y,z) = -T(x,\varphi y,\varphi z),\ t = 0$.

2. Almost complex manifolds with Norden metric

Let us consider an almost complex manifold with Norden metric or an *almost complex Norden manifold* (M', J, g'), i.e.

$$J^2x = -x, \qquad g'(Jx, Jy) = -g'(x,y) \tag{3}$$

for all differentiable vector fields x, y on M'. It is $2n$-dimensional. The *associated metric* $\widetilde{g}'$ of g' on M' defined by $\widetilde{g}'(x,y) = g'(x, Jy)$ is also a Norden metric. The signature of both the metrics is necessarily (n,n).

These manifolds are known as almost complex manifolds with Norden metric [5, 31, 32], almost complex manifolds with B-metric [6, 8] or almost complex manifolds with complex Riemannian metric [2, 7, 13, 25]. Their structure group is $GL(n,\mathbb{C}) \cap O(n,n)$.

Further in this section, x, y, z, w will stand for arbitrary differentiable vector fields on M' (or vectors in the tangent space of M' at an arbitrary point of M'). Moreover, let $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ be an arbitrary basis of the tangent space of M' at any point of M' and g'^{ij} be the corresponding components of the inverse matrix of g'.

The fundamental tensor F' of type $(0,3)$ on M' is defined by $F'(x,y,z) = g'\big((\nabla'_x J)\,y, z\big)$, where ∇' is the Levi-Civita connection of g', and F' has the following properties: [11]

$$F'(x,y,z) = F'(x,z,y) = F'(x,Jy,Jz). \tag{4}$$

The corresponding Lee form θ' is defined by $\theta'(z) = g'^{ij}F'(e_i,e_j,z)$. The associated trace with respect to the metric $\widetilde{g}'$ is defined by $\widetilde{\theta}'(z) = \widetilde{g}'^{ij}F'(e_i,e_j,z)$, which implies the relation $\widetilde{\theta}'(z) = \theta'(Jz)$ because of $\widetilde{g}'^{ij}F'(e_i,e_j,z) = -g'^{ij}F'(e_i,Je_j,z) = g'^{ij}F'(e_i,e_j,Jz)$.

In [4], the considered manifolds are classified into three basic classes $\mathcal{W}_i$ $(i = 1,2,3)$ with respect to F'. All classes are determined as follows:

$$\begin{array}{rl}
\mathcal{W}_0: & F'(x,y,z) = 0;\\
\mathcal{W}_1: & F'(x,y,z) = \frac{1}{2n}\big\{g'(x,y)\theta'(z) + g'(x,Jy)\theta'(Jz)\big\}_{(y\leftrightarrow z)};\\
\mathcal{W}_2: & \mathop{\mathfrak{S}}\limits_{x,y,z} F'(x,y,Jz) = 0, \quad \theta' = 0;\\
\mathcal{W}_3: & \mathop{\mathfrak{S}}\limits_{x,y,z} F'(x,y,z) = 0;\\
\mathcal{W}_1\oplus\mathcal{W}_2: & \mathop{\mathfrak{S}}\limits_{x,y,z} F'(x,y,Jz) = 0;\\
\mathcal{W}_1\oplus\mathcal{W}_3: & \mathop{\mathfrak{S}}\limits_{x,y,z} F'(x,y,z) = \frac{1}{n}\mathop{\mathfrak{S}}\limits_{x,y,z}\big\{g'(x,y)\theta'(z) + g'(x,Jy)\theta'(Jz)\big\};\\
\mathcal{W}_2\oplus\mathcal{W}_3: & \theta' = 0;\\
\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3: & \text{no conditions.}
\end{array} \tag{5}$$

The class $\mathcal{W}_0$ of the *Kähler manifolds with Norden metric* belongs to any other class.

Let R' be the curvature tensor of ∇', i.e. $R' = [\nabla'\ ,\nabla'\ ] - \nabla'_{[\ ,\ ]}$ and the corresponding $(0,4)$-tensor is determined by $R'(x,y,z,w) = g'(R'(x,y)z,w)$. The Ricci tensor ρ' and the scalar curvature τ' are defined as usual by $\rho'(y,z) = g'^{ij}R'(e_i,y,z,e_j)$ and $\tau' = g'^{ij}\rho'(e_i,e_j)$.

A tensor L of type (0,4) having the properties $L(x,y,z,w) = -L(y,x,z,w) = -L(x,y,w,z)$, $\mathop{\mathfrak{S}}\limits_{x,y,z} L(x,y,z,w) = 0$ is called a *curvature-like tensor*. Moreover, if the curvature-like tensor L has the property $L(x,y,Jz,Jw) = -L(x,y,z,w)$, it is called a *Kähler tensor* [6].

2.1. *The pair of the Nijenhuis tensors*

As it is well known, the Nijenhuis tensor N' of the almost complex structure J is defined by

$$N'(x,y) := [J,J](x,y) = [Jx,Jy] - [x,y] - J[Jx,y] - J[x,Jy]\,. \tag{6}$$

Besides it, we define the following symmetric (1,2)-tensor $\widehat{N}'$ in analogy to (6) by

$$\widehat{N}'(x,y) = \{J,J\}(x,y) = \{Jx,Jy\} - \{x,y\} - J\{Jx,y\} - J\{x,Jy\},$$

where the symmetric braces $\{x,y\} = \nabla_x y + \nabla_y x$ are used instead of the antisymmetric brackets $[x,y] = \nabla_x y - \nabla_y x$. The tensor $\widehat{N}'$ we also call the *associated Nijenhuis tensor* of the almost complex structure.

The pair of the Nijenhuis tensors N' and $\widehat{N}'$ plays a fundamental role in the topic of natural connections (i.e. J and g' are parallel with respect to them) on an almost complex Norden manifold. The torsions and the potentials of these connections are expressed by these two tensors. By this reason we characterize the classes of the considered manifolds in terms of N' and $\widehat{N}'$.

As it is known from [4], the class $\mathcal{W}_3$ of the *quasi-Kähler manifolds with Norden metric* is the only basic class of the considered manifolds with non-integrable almost complex structure J, because N' is non-zero there. Moreover, this class is determined by the condition $\widehat{N}' = 0$. The class $\mathcal{W}_1\oplus\mathcal{W}_2$ of the *(integrable* *almost) complex manifolds with Norden metric* is characterized by $N' = 0$ and $\widehat{N}' \neq 0$. Additionally, the basic classes $\mathcal{W}_1$ and $\mathcal{W}_2$ are distinguish from each other according to the Lee form θ': for $\mathcal{W}_1$ the tensor F' is expressed explicitly by the metric and the Lee form, i.e. $\theta' \neq 0$; whereas for $\mathcal{W}_2$ it is valid $\theta' = 0$.

The corresponding tensors of type (0,3) are denoted by the same letter, $N'(x,y,z) = g'(N'(x,y),z)$, $\widehat{N}'(x,y,z) = g'(\widehat{N}'(x,y),z)$. Both tensors N' and $\widehat{N}'$ can be expressed in terms of F' as follows: [4]

$$N'(x,y,z) = F'(x,Jy,z) - F'(y,Jx,z) + F'(Jx,y,z) - F'(Jy,x,z), \tag{7}$$

$$\widehat{N}'(x,y,z) = F'(x,Jy,z) + F'(y,Jx,z) + F'(Jx,y,z) + F'(Jy,x,z). \tag{8}$$

The tensor $\widehat{N}'$ coincides with the tensor $\widetilde{N}'$ introduced in [4] by an equivalent equality of (8).

By virtue of (3), (4), (7) and (8), we get the following properties of N' and $\widehat{N}'$:

$$N'(x,y,z) = N'(x,Jy,Jz) = N'(Jx,y,Jz) = -N'(Jx,Jy,z), \quad N'(Jx,y,z) = N'(x,Jy,z) = -N'(x,y,Jz); \tag{9}$$

$$\widehat{N}'(x,y,z) = \widehat{N}'(x,Jy,Jz) = \widehat{N}'(Jx,y,Jz) = -\widehat{N}'(Jx,Jy,z), \quad \widehat{N}'(Jx,y,z) = \widehat{N}'(x,Jy,z) = -\widehat{N}'(x,y,Jz). \tag{10}$$

Theorem 2.1. *The fundamental tensor F' of an almost complex Norden manifold (M',J,g') is expressed in terms of the Nijenhuis tensors N' and $\widehat{N}'$ by the formula*

$$F'(x,y,z) = -\frac{1}{4}\left\{N'(Jx,y,z) + N'(Jx,z,y) + \widehat{N}'(Jx,y,z) + \widehat{N}'(Jx,z,y)\right\}. \tag{11}$$

Proof. Taking the sum of (7) and (8), we obtain

$$F'(Jx,y,z) + F'(x,Jy,z) = \frac{1}{2}\left\{N'(x,y,z) + \widehat{N}'(x,y,z)\right\}. \tag{12}$$

The identities (3) and (4) imply

$$F'(x,z,Jy) = -F'(x,y,Jz). \tag{13}$$

A suitable combination of (12) and (13) yields

$$F'(Jx,y,z) = \frac{1}{4}\left\{N'(x,y,z) + N'(x,z,y) + \widehat{N}'(x,y,z) + \widehat{N}'(x,z,y)\right\}. \tag{14}$$

Applying (3) to (14), we obtain the stated formula. □

As direct corollaries of Theorem 2.1 we have:

$$\mathcal{W}_1 \oplus \mathcal{W}_2 : \ F'(x,y,z) = -\frac{1}{4}\left\{\widehat{N}'(Jx,y,z) + \widehat{N}'(Jx,z,y)\right\}, \quad \mathcal{W}_3 : \ F'(x,y,z) = -\frac{1}{4}\left\{N'(Jx,y,z) + N'(Jx,z,y)\right\}. \tag{15}$$

According to Theorem 2.1, we obtain the following relation for the corresponding traces:

$$\theta' = \frac{1}{4}\widehat{\nu}' \circ J, \tag{16}$$

where $\widehat{\nu}'(z) = g'^{ij}\widehat{N}'(e_i,e_j,z)$. For the traces with respect to the associated metric $\widetilde{g}'$ of F' and $\widehat{N}'$, i.e. $\widetilde{\theta}'(z) = \widetilde{g}'^{ij}F'(e_i,e_j,z)$ and $\widetilde{\widehat{\nu}}'(z) = \widetilde{g}'^{ij}\widehat{N}'(e_i,e_j,z)$, we have $\widetilde{\theta}' = -\frac{1}{4}\widehat{\nu} = \theta' \circ J$ and $\widetilde{\widehat{\nu}}' = 4\theta' = \widehat{\nu}' \circ J$, respectively.

Then, bearing in mind (5) and the subsequent comments on the pair of the Nijenhuis tensors, from Theorem 2.1 and (16) we obtain immediately the following

Theorem 2.2. *The classes of almost complex Norden manifolds are characterized by the Nijenhuis tensors N' and $\widehat{N}'$ as follows:*

$$\begin{array}{rll}
\mathcal{W}_0: & N'=0, & \widehat{N}'=0;\\
\mathcal{W}_1: & N'=0, & \widehat{N}'=\frac{1}{2n}\left\{\widehat{\nu}'\otimes g'+\widetilde{\widehat{\nu}'}\otimes\widetilde{g}'\right\};\\
\mathcal{W}_2: & N'=0, & \widehat{\nu}'=0;\\
\mathcal{W}_3: & & \widehat{N}'=0;\\
\mathcal{W}_1\oplus\mathcal{W}_2: & N'=0; & \\
\mathcal{W}_1\oplus\mathcal{W}_3: & & \widehat{N}'=\frac{1}{2n}\left\{\widehat{\nu}'\otimes g'+\widetilde{\widehat{\nu}'}\otimes\widetilde{g}'\right\};\\
\mathcal{W}_2\oplus\mathcal{W}_3: & & \widehat{\nu}'=0;\\
\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3: & \text{no conditions.} &
\end{array} \tag{17}$$

2.2. Natural connections on an almost complex Norden manifold

Let ∇^* be a linear connection with a torsion T^* and a potential Q^* with respect to ∇', i.e.

$$T^*(x,y)=\nabla^*_x y-\nabla^*_y x-[x,y],\quad Q^*(x,y)=\nabla^*_x y-\nabla'_x y.$$

The corresponding (0,3)-tensors are defined by $T^*(x,y,z)=g'(T^*(x,y),z)$, $Q^*(x,y,z)=g'(Q^*(x,y),z)$. These tensors have the same mutual relations as in (1) and (2).

In [8], it is given a partial decomposition of the space $\mathcal{T}$ of all torsion (0,3)-tensors T (i.e. satisfying $T(x,y,z)=-T(y,x,z)$) on an almost complex Norden manifold (M',J,g'): $\mathcal{T}=\mathcal{T}_1\oplus\mathcal{T}_2\oplus\mathcal{T}_3\oplus\mathcal{T}_4$, where $\mathcal{T}_i$ $(i=1,2,3,4)$ are invariant orthogonal subspaces with respect to the structure group $GL(n,\mathbb{C})\cap O(n,n)$:

$$\begin{array}{ll}
\mathcal{T}_1: & T(x,y,z)=-T(Jx,Jy,z)=-T(Jx,y,Jz);\\
\mathcal{T}_2: & T(x,y,z)=-T(Jx,Jy,z)=T(Jx,y,Jz);\\
\mathcal{T}_3: & T(x,y,z)=T(Jx,Jy,z),\quad \underset{x,y,z}{\mathfrak{S}} T(x,y,z)=0,\\
\mathcal{T}_4: & T(x,y,z)=T(Jx,Jy,z),\quad \underset{x,y,z}{\mathfrak{S}} T(Jx,y,z)=0.
\end{array}$$

Moreover, in [8] there are explicitly given the components T_i of $T\in\mathcal{T}$ in $\mathcal{T}_i$ $(i=1,2,3,4)$.

A linear connection ∇^* on an almost complex manifold with Norden metric (M',J,g') is called a *natural connection* if $\nabla^*J=\nabla^*g'=0$. These conditions are equivalent to $\nabla^*g'=\nabla^*\widetilde{g}'=0$. The connection ∇^* is natural if and only if the following conditions for its potential Q^* are valid:

$$F'(x,y,z)=Q^*(x,y,Jz)-Q^*(x,Jy,z),\quad Q^*(x,y,z)=-Q^*(x,z,y). \tag{18}$$

In terms of the components T_i, a linear connection with torsion T on (M',J,g') is natural if and only if

$$T_2(x,y,z)=\frac{1}{4}N'(x,y,z),\quad T_3(x,y,z)=\frac{1}{8}\left\{\widehat{N}'(z,y,x)-\widehat{N}'(z,x,y)\right\}.$$

The former condition is given in [8] whereas the latter one follows immediately by (1), (2), (8) and (18).

2.3. The B-connection and the canonical connection

In [6], it is introduced the *B-connection* $\dot{\nabla}'$ only for the manifolds from the class $\mathcal{W}_1$ by

$$\dot{\nabla}'_x y=\nabla'_x y-\frac{1}{2}J\left(\nabla'_x J\right)y. \tag{19}$$

Obviously, the B-connection is a natural connection on (M',J,g') and it exists in any class of the considered manifolds. Only on a $\mathcal{W}_0$-manifold, the B-connection coincides with the Levi-Civita connection.

By virtue of (1), (9), (10), (11), from (19) we express the torsion of the B-connection as follows:

$$\dot{T}'(x,y,z)=\frac{1}{8}\left\{N'(x,y,z)+\underset{x,y,z}{\mathfrak{S}} N'(x,y,z)+\widehat{N}'(z,y,x)-\widehat{N}'(z,x,y)\right\}. \tag{20}$$

A natural connection with torsion $\ddot{T}'$ on an almost complex manifold with Norden metric (M', J, g') is called a *canonical connection* if $\ddot{T}'$ satisfies the following condition [8]

$$\ddot{T}'(x,y,z) + \ddot{T}'(y,z,x) - \ddot{T}'(Jx,y,Jz) - \ddot{T}'(y,Jz,Jx) = 0. \tag{21}$$

In [8] it is shown that (21) is equivalent to the condition $\ddot{T}'_1 = \ddot{T}'_4 = 0$, i.e. $\ddot{T}' \in \mathcal{T}_2 \oplus \mathcal{T}_3$. Moreover, there it is proved that on every almost complex Norden manifold there exists a unique canonical connection $\ddot{\nabla}'$. We express its torsion in terms of N' and $\widehat{N}'$ as follows

$$\ddot{T}'(x,y,z) = \frac{1}{4}N'(x,y,z) + \frac{1}{8}\Big\{\widehat{N}'(z,y,x) - \widehat{N}'(z,x,y)\Big\}. \tag{22}$$

Taking into account (22) and (20), it is easy to conclude that $\ddot{\nabla}' \equiv \dot{\nabla}'$ is valid if and only if the condition $N' = \mathfrak{S}N'$ holds which is equivalent to $N' = 0$. In other words, on a complex Norden manifold, i.e. $(M', J, g') \in \mathcal{W}_1 \oplus \mathcal{W}_2$, the canonical connection and the B-connection coincide.

Now, let (M', J, g') be in the class $\mathcal{W}_1$. This is the class of the conformally equivalent manifolds of the Kähler manifold with Norden metric. The conformal equivalence is made with respect to the general conformal transformations of the metric g' defined by

$$\overline{g}' = e^{2u}\left\{\cos 2v\, g' + \sin 2v\, \widetilde{g}'\right\}, \tag{23}$$

where u and v are differentiable functions on M' [6]. For $v = 0$ they are restricted to the usual conformal transformations. The manifold $(M', J, \overline{g}')$ is again an almost complex Norden manifold. An important subgroup of the general group C of the conformal transformations (23) is the group C_0 of the *holomorphic conformal transformations*, defined by the condition: $u + iv$ is a holomorphic function, i.e. $\mathrm{d}u = \mathrm{d}v \circ J$. Then torsion of the canonical connection is an invariant of C_0, i.e. the relation $\overline{\ddot{T}'}(x,y) = \ddot{T}'(x,y)$ holds with respect to any transformation of C_0. There are proved that the curvature tensor of the canonical connection is a Kähler tensor if and only if $(M', J, g') \in \mathcal{W}_1^0$, i.e. a manifold in $\mathcal{W}_1$ with closed forms θ' and $\theta' \circ J$. Moreover, there are studied conformal invariants of the canonical connection in $\mathcal{W}_1^0$.

Bearing in mind the conformal invariance of both the basic classes and the torsion $\ddot{T}'$ of the canonical connection, the conditions for $\ddot{T}'$ are used in [8] for other characteristics of all classes of the almost complex Norden manifolds as follows:

$$\begin{array}{rl}
\mathcal{W}_0: & \ddot{T}'(x,y) = 0;\\
\mathcal{W}_1: & \ddot{T}'(x,y) = \frac{1}{2n}\left\{\ddot{t}'(x)y - \ddot{t}'(y)x + \ddot{t}'(Jx)Jy - \ddot{t}'(Jy)Jx\right\};\\
\mathcal{W}_2: & \ddot{T}'(x,y) = \ddot{T}'(Jx,Jy), \quad \ddot{t}' = 0;\\
\mathcal{W}_3: & \ddot{T}'(Jx,y) = -J\ddot{T}'(x,y);\\
\mathcal{W}_1 \oplus \mathcal{W}_2: & \ddot{T}'(x,y) = \ddot{T}'(Jx,Jy), \quad \underset{x,y,z}{\mathfrak{S}} \ddot{T}'(x,y,z) = 0;\\
\mathcal{W}_1 \oplus \mathcal{W}_3: & \ddot{T}'(Jx,y) + J\ddot{T}'(x,y) = \frac{1}{n}\Big\{\ddot{t}'(Jy)x - \ddot{t}'(y)Jx\Big\};\\
\mathcal{W}_2 \oplus \mathcal{W}_3: & \ddot{t}' = 0;\\
\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3: & \text{no conditions,}
\end{array} \tag{24}$$

where $\ddot{t}'(x) = g'^{ij}\ddot{T}'(x, e_i, e_j)$. The special class $\mathcal{W}_0$ is characterized by the condition $\ddot{T}'(x,y) = 0$, i.e. there $\ddot{\nabla}' \equiv \nabla'$ holds.

The classes of the almost complex Norden manifolds are determined with respect to the Nijenhuis tensors in (17), the same classes are characterized by conditions for the torsion of the canonical connection in (24). By virtue of these results we obtain the following

Theorem 2.3. *The classes of the almost complex Norden manifolds $M = (M, \varphi, \xi, \eta, g)$ are characterized by an* *expression of the torsion $\ddot{T}'$ of the canonical connection in terms of the Nijenhuis tensors N and $\widehat{N}$ as follows:*

$$
\begin{array}{rl}
\mathcal{W}_1: & \ddot{T}'(x,y,z) = \frac{1}{16n}\Big\{\widehat{\nu}'(x)g'(y,z) + \widehat{\nu}'(Jx)g'(y,Jz)\Big\}_{[x\leftrightarrow y]};\\
\mathcal{W}_2: & \ddot{T}'(x,y,z) = \frac{1}{8}\Big\{\widehat{N}'(z,y,x) - \widehat{N}'(z,x,y)\Big\}, \quad \ddot{t}' = \widehat{\nu}' = 0;\\
\mathcal{W}_3: & \ddot{T}'(x,y,z) = \frac{1}{4}N'(x,y,z);\\
\mathcal{W}_1\oplus\mathcal{W}_2: & \ddot{T}'(x,y,z) = \frac{1}{8}\Big\{\widehat{N}'(z,y,x) - \widehat{N}'(z,x,y)\Big\};\\
\mathcal{W}_1\oplus\mathcal{W}_3: & \ddot{T}'(x,y,z) = \frac{1}{4}N'(x,y,z) + \frac{1}{16n}\Big\{\widehat{\nu}'(x)g'(y,z) + \widehat{\nu}'(Jx)g'(y,Jz)\Big\}_{[x\leftrightarrow y]};\\
\mathcal{W}_2\oplus\mathcal{W}_3: & \ddot{T}'(x,y,z) = \frac{1}{4}N'(x,y,z) + \frac{1}{8}\Big\{\widehat{N}'(z,y,x) - \widehat{N}'(z,x,y)\Big\}, \quad \ddot{t}' = \widehat{\nu}' = 0.
\end{array}
\tag{25}
$$

The special class $\mathcal{W}_0$ is characterized by $\ddot{T}' = 0$ and the whole class $\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$ — by (22) only.

Moreover, bearing in mind the classifications with respect to the tensor F' and the torsion $\ddot{T}'$ in [4] and [8], respectively, we have:

- *(i) $M \in \mathcal{W}_1\oplus\mathcal{W}_2$ if and only if $\ddot{T}' \in \mathcal{T}_3$;*
- *(ii) $M \in \mathcal{W}_1$ if and only if $\ddot{T}' \in \mathcal{T}_3^1$, where $\mathcal{T}_3^1$ is the subclass of $\mathcal{T}_3$ with the vectorial torsions[1)];*
- *(iii) $M \in \mathcal{W}_2$ if and only if $\ddot{T}' \in \mathcal{T}_3^0$, where $\mathcal{T}_3^0$ is the subclass of $\mathcal{T}_3$ with $\ddot{t}' = 0$;*
- *(iv) $M \in \mathcal{W}_3$ if and only if $\ddot{T}' \in \mathcal{T}_2$;*
- *(v) $M \in \mathcal{W}_1\oplus\mathcal{W}_3$ if and only if $\ddot{T}' \in \mathcal{T}_2\oplus\mathcal{T}_3^1$;*
- *(vi) $M \in \mathcal{W}_2\oplus\mathcal{W}_3$ if and only if $\ddot{T}' \in \mathcal{T}_2\oplus\mathcal{T}_3^0$;*
- *(vii) $M \in \mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$ if and only if $\ddot{T}' \in \mathcal{T}_2\oplus\mathcal{T}_3$.*

Proof. Let (M',J,g') be a complex Norden manifold, i.e. $(M',J,g') \in \mathcal{W}_1\oplus\mathcal{W}_2$. According to (22) and $N' = 0$ in this case, we have $\ddot{T}' = \ddot{T}_3'$, i.e. $\ddot{T}' \in \mathcal{T}_3$ and the expression $\ddot{T}'(x,y,z) = \frac{1}{8}\Big\{\widehat{N}'(z,y,x) - \widehat{N}'(z,x,y)\Big\}$ is obtained. Applying (17) to the latter equality, we determine the basic classes $\mathcal{W}_1$ and $\mathcal{W}_2$ as is given in (25) and the corresponding subclasses $\mathcal{T}_3^1$ and $\mathcal{T}_3^0$, respectively. Taking into account the relation between the corresponding traces $\widehat{\nu}' = 8\ddot{t}'$, which is a consequence of the equality for $\mathcal{W}_1\oplus\mathcal{W}_2$, we obtain the characterization for these two basic classes in (24).

Let (M',J,g') be a quasi-Kähler manifold with Norden metric, i.e. $(M',J,g') \in \mathcal{W}_3$. By virtue of (22) and $\widehat{N}' = 0$ for such a manifold, we have $\ddot{T}' = \ddot{T}_2'$, i.e. $\ddot{T}' \in \mathcal{T}_2$ and therefore we give $\ddot{T}' = \frac{1}{4}N'$. Obviously, the form of $\ddot{T}'$ in the latter equality satisfies the condition for $\mathcal{W}_3$ in (24).

In a similar way we get for the rest classes $\mathcal{W}_1\oplus\mathcal{W}_3$ and $\mathcal{W}_2\oplus\mathcal{W}_3$. The conditions of these two classes, given in (24), are consequences of the corresponding equalities in (25). The case of the whole class $\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$ was discussed above. □

The canonical connections on quasi-Kähler manifolds with Norden metric are considered in more details in [28]. There are given the following formulae for the potential $\ddot{Q}'$ and the torsion $\ddot{T}'$ on a $\mathcal{W}_3$-manifold:

$$
\ddot{Q}'(x,y) = \frac{1}{4}\left\{\left(\nabla'_y J\right)Jx - \left(\nabla'_{Jy}J\right)x + 2\left(\nabla'_x J\right)Jy\right\}, \quad \ddot{T}'(x,y) = \frac{1}{2}\left\{\left(\nabla'_x J\right)Jy + \left(\nabla'_{Jx}J\right)y\right\}.
$$

Moreover, some properties for the curvature and the torsion of the canonical connection are obtained.

[1)] A *vectorial torsion* is a torsion which is essentially defined by some vector field on the manifold and its metrics.

2.4. The KT-connection

In [27], it is proved that a natural connection $\dddot{\nabla}'$ with totally skew-symmetric torsion, called a *KT-connection*, exists on an almost complex Norden manifold (M', J, g') if and only if (M', J, g') belongs to $\mathcal{W}_3$, i.e. the manifold is quasi-Kählerian with Norden metric. Moreover, the KT-connection is unique and it is determined by its potential

$$\dddot{Q}'(x,y,z) = -\frac{1}{4}\underset{x,y,z}{\mathfrak{S}} F'(x,y,Jz). \tag{26}$$

As mentioned above, the canonical connection and the B-connection coincide on $(M', J, g') \in \mathcal{W}_1 \oplus \mathcal{W}_2$ whereas the KT-connection does not exist there.

The following natural connections on (M', J, g') are studied on a quasi-Kähler manifold with Norden metric: the B-connection $\dot{\nabla}'$ ([26]), the canonical connection $\ddot{\nabla}'$ ([28]) and the KT-connection $\dddot{\nabla}'$ ([27]).

From the relations (22) and (20) for a $\mathcal{W}_3$-manifold follow

$$\dot{T}'(x,y,z) = \frac{1}{8}\Big\{N'(x,y,z) + \underset{x,y,z}{\mathfrak{S}} N'(x,y,z)\Big\}, \quad \ddot{T}'(x,y,z) = \frac{1}{4}N'(x,y,z). \tag{27}$$

The equalities (1) and (26) yield $\dddot{T}'(x,y,z) = -\frac{1}{2}\underset{x,y,z}{\mathfrak{S}} F'(x,y,Jz)$, which by (15) for $\mathcal{W}_3$ and (9) implies

$$\dddot{T}'(x,y,z) = \frac{1}{4}\underset{x,y,z}{\mathfrak{S}} N'(x,y,z). \tag{28}$$

Then from (27) and (28) we have the relation $\dot{T}' = \frac{1}{2}\left(\ddot{T}' + \dddot{T}'\right)$, which by (2) is equivalent to $\dot{Q}' = \frac{1}{2}\left(\ddot{Q}' + \dddot{Q}'\right)$. Therefore, as it is shown in [28], the B-connection is the *average* connection for the canonical connection and the KT-connection on a quasi-Kähler manifold with Norden metric , i.e. $\dot{\nabla}' = \frac{1}{2}\left(\ddot{\nabla}' + \dddot{\nabla}'\right)$.

3. Almost contact manifolds with B-metric

Let (M, φ, ξ, η) be an almost contact manifold, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ and its dual 1-form η such that the following relations are valid:

$$\varphi\xi = 0, \quad \varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \tag{29}$$

Later on, let us equip (M, φ, ξ, η) with a pseudo-Riemannian metric g of signature $(n+1, n)$ determined by

$$g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y) \tag{30}$$

for arbitrary differentiable vector fields x, y on M. Then $(M, \varphi, \xi, \eta, g)$ is called an almost contact manifold with B-metric or an *almost contact B-metric manifold*. The associated metric $\widetilde{g}$ of g on M is defined by the equality $\widetilde{g}(x,y) = g(x, \varphi y) + \eta(x)\eta(y)$. Both metrics g and $\widetilde{g}$ are necessarily of signature $(n+1, n)$. The manifold $(M, \varphi, \xi, \eta, \widetilde{g})$ is also an almost contact B-metric manifold. [9]

Let us remark that the $2n$-dimensional contact distribution $H = \ker(\eta)$, generated by the contact 1-form η, can be considered as the horizontal distribution of the sub-Riemannian manifold M. Then H is endowed with an almost complex structure determined as $\varphi|_H$ – the restriction of φ on H, as well as a Norden metric $g|_H$, i.e. $g|_H(\varphi|_H\cdot, \varphi|_H\cdot) = -g|_H(\cdot,\cdot)$. Moreover, H can be considered as an n-dimensional complex Riemannian manifold with a complex Riemannian metric $g^{\mathbb{C}} = g|_H + i\widetilde{g}|_H$ [7]. By this reason we refer to these manifolds as *almost contact complex Riemannian manifolds*. They are investigated and studied in [9, 17, 19–21, 23, 24, 30]. The structure group of these manifolds is $(GL(n,\mathbb{C}) \cap O(n,n)) \times I_1$.

Further in this section, x, y, z will stand for arbitrary differentiable vector fields on M (or vectors in the tangent space of M at an arbitrary point of M). Moreover, let $\{e_i; \xi\}_{i=1}^{2n}$ denote an arbitrary basis of the tangent space of M at an arbitrary point in M and g^{ij} be the corresponding components of the inverse matrix of g.

The fundamental tensor F of type (0,3) on (M,φ,ξ,η,g) is defined by $F(x,y,z)=g\big((\nabla_x\varphi)\,y,z\big)$, where ∇ is the Levi-Civita connection for g and the following properties are valid: [9]

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi). \tag{31}$$

The relations of the covariant derivatives $\nabla\xi$ and $\nabla\eta$ with F are: $(\nabla_x\eta)\,y=g\,(\nabla_x\xi,y)=F(x,\varphi y,\xi)$.

The following 1-forms, called Lee forms, are associated with F:

$$\theta(z)=g^{ij}F(e_i,e_j,z),\quad \theta^*(z)=g^{ij}F(e_i,\varphi e_j,z),\quad \omega(z)=F(\xi,\xi,z).$$

Obviously, the equalities $\theta^*\circ\varphi=-\theta\circ\varphi^2$ and $\omega(\xi)=0$ are valid. For the corresponding traces $\widetilde{\theta}$ and $\widetilde{\theta^*}$ with respect to $\widetilde{g}$ we have $\widetilde{\theta}=-\theta^*$ and $\widetilde{\theta^*}=\theta$.

A classification with respect to F of the almost contact B-metric manifolds is given in [9]. This classification includes eleven basic classes $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_{11}$. Their intersection is the special class $\mathcal{F}_0: F(x,y,z)=0$.

Further, we use the following characteristic conditions of the basic classes: [9, 18]

$$\begin{array}{rl}
\mathcal{F}_1: & F(x,y,z)=\frac{1}{2n}\big\{g(x,\varphi y)\theta(\varphi z)+g(\varphi x,\varphi y)\theta(\varphi^2 z)\big\}_{(y\leftrightarrow z)};\\
\mathcal{F}_2: & F(\xi,y,z)=F(x,\xi,z)=0,\quad \mathop{\mathfrak{S}}\limits_{x,y,z}F(x,y,\varphi z)=0,\quad \theta=0;\\
\mathcal{F}_3: & F(\xi,y,z)=F(x,\xi,z)=0,\quad \mathop{\mathfrak{S}}\limits_{x,y,z}F(x,y,z)=0;\\
\mathcal{F}_4: & F(x,y,z)=-\frac{\theta(\xi)}{2n}\big\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\big\};\\
\mathcal{F}_5: & F(x,y,z)=-\frac{\theta^*(\xi)}{2n}\big\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\big\};\\
\mathcal{F}_{6/7}: & F(x,y,z)=F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\quad F(x,y,\xi)=\pm F(y,x,\xi)=-F(\varphi x,\varphi y,\xi),\quad \theta=\theta^*=0;\\
\mathcal{F}_{8/9}: & F(x,y,z)=F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\quad F(x,y,\xi)=\pm F(y,x,\xi)=F(\varphi x,\varphi y,\xi);\\
\mathcal{F}_{10}: & F(x,y,z)=F(\xi,\varphi y,\varphi z)\eta(x);\\
\mathcal{F}_{11}: & F(x,y,z)=\eta(x)\left\{\eta(y)\omega(z)+\eta(z)\omega(y)\right\}.
\end{array}$$

3.1. The pair of the Nijenhuis tensors

An almost contact structure (φ,ξ,η) on M is called *normal* and respectively (M,φ,ξ,η) is a *normal almost contact manifold* if the corresponding almost complex structure J' generated on $M'=M\times\mathbb{R}$ is integrable [34]. The almost contact structure is normal if and only if the Nijenhuis tensor of (φ,ξ,η) is zero [1].

The Nijenhuis tensor N of the almost contact structure is defined by $N:=[\varphi,\varphi]+\mathrm{d}\eta\otimes\xi$, where $[\varphi,\varphi](x,y)=[\varphi x,\varphi y]+\varphi^2[x,y]-\varphi[\varphi x,y]-\varphi[x,\varphi y]$ and $\mathrm{d}\eta$ is the exterior derivative of η.

In [24], it is defined the symmetric (1,2)-tensor $\widehat{N}$ for a (φ,ξ,η)-structure by $\widehat{N}=\{\varphi,\varphi\}+(\mathcal{L}_\xi g)\otimes\xi$, where $\mathcal{L}$ denotes the Lie derivative and $\{\varphi,\varphi\}$ is given by $\{\varphi,\varphi\}(x,y)=\{\varphi x,\varphi y\}+\varphi^2\{x,y\}-\varphi\{\varphi x,y\}-\varphi\{x,\varphi y\}$ for $\{x,y\}=\nabla_x y+\nabla_y x$. The tensor $\widehat{N}$ is also called the *associated Nijenhuis tensor* for (φ,ξ,η).

Obviously, N is antisymmetric and $\widehat{N}$ is symmetric, i.e. $N(x,y)=-N(y,x)$ and $\widehat{N}(x,y)=\widehat{N}(y,x)$.

The Nijenhuis tensors N and $\widehat{N}$ play a fundamental role in natural connections (i.e. such connections that the tensors of the structure (φ,ξ,η,g) are parallel with respect to them) on an almost contact B-metric manifold. The torsions and the potentials of these connections are expressed by these two tensors. By this reason we characterize the classes of the considered manifolds in terms of N and $\widehat{N}$.

The corresponding tensors of type (0,3) are denoted by the same letters, $N(x,y,z)=g(N(x,y),z)$, $\widehat{N}(x,y,z)=g(\widehat{N}(x,y),z)$. Both tensors N and $\widehat{N}$ are expressed in terms of F as follows [24]

$$N(x,y,z)=\big\{F(\varphi x,y,z)-F(x,y,\varphi z)+\eta(z)F(x,\varphi y,\xi)\big\}_{[x\leftrightarrow y]}, \tag{32}$$

$$\widehat{N}(x,y,z)=\big\{F(\varphi x,y,z)-F(x,y,\varphi z)+\eta(z)F(x,\varphi y,\xi)\big\}_{(x\leftrightarrow y)}. \tag{33}$$

Bearing in mind (29), (30) and (31), from (32) and (33) we obtain the following properties of the Nijenhuis tensors on an arbitrary almost contact B-metric manifold:

$$
\begin{aligned}
&N(x,\varphi y,\varphi z)=N(x,\varphi^2 y,\varphi^2 z), \quad N(\varphi x,y,\varphi z)=N(\varphi^2 x,y,\varphi^2 z), \quad N(\varphi x,\varphi y,z)=-N(\varphi^2 x,\varphi^2 y,z),\\
&\widehat{N}(x,\varphi y,\varphi z)=\widehat{N}(x,\varphi^2 y,\varphi^2 z), \quad \widehat{N}(\varphi x,y,\varphi z)=\widehat{N}(\varphi^2 x,y,\varphi^2 z), \quad \widehat{N}(\varphi x,\varphi y,z)=-\widehat{N}(\varphi^2 x,\varphi^2 y,z),\\
&N(\xi,\varphi y,\varphi z)+N(\xi,\varphi z,\varphi y)+\widehat{N}(\xi,\varphi y,\varphi z)+\widehat{N}(\xi,\varphi z,\varphi y)=0.
\end{aligned}
$$

It is known that the class of the normal almost contact B-metric manifolds, i.e. $N=0$, is $\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6$. According to [24], the class of the almost contact B-metric manifolds with $\widehat{N}=0$ is $\mathcal{F}_3\oplus\mathcal{F}_7$. The latter two statements follow from [23] and [24], where the following form of the Nijenhuis tensors for each of the basic classes $\mathcal{F}_i$ $(i=1,2,\dots,11)$ of (M,φ,ξ,η,g) is given:

$$
\begin{array}{lll}
\mathcal{F}_1: & N(x,y)=0, & \widehat{N}(x,y)=\frac{2}{n}\left\{g(\varphi x,\varphi y)\varphi\theta^\sharp+g(x,\varphi y)\theta^\sharp\right\};\\
\mathcal{F}_2: & N(x,y)=0, & \widehat{N}(x,y)=2\left\{\left(\nabla_{\varphi x}\varphi\right)y-\varphi\left(\nabla_x\varphi\right)y\right\};\\
\mathcal{F}_3: & N(x,y)=2\left\{\left(\nabla_{\varphi x}\varphi\right)y-\varphi\left(\nabla_x\varphi\right)y\right\}, & \widehat{N}(x,y)=0;\\
\mathcal{F}_4: & N(x,y)=0, & \widehat{N}(x,y)=\frac{2}{n}\theta(\xi)g(x,\varphi y)\xi;\\
\mathcal{F}_5: & N(x,y)=0, & \widehat{N}(x,y)=-\frac{2}{n}\theta^*(\xi)g(\varphi x,\varphi y)\xi;\\
\mathcal{F}_6: & N(x,y)=0, & \widehat{N}(x,y)=4\left(\nabla_x\eta\right)y\,\xi;\\
\mathcal{F}_7: & N(x,y)=4\left(\nabla_x\eta\right)y\,\xi, & \widehat{N}(x,y)=0;\\
\mathcal{F}_8: & N(x,y)=2\left\{\eta(x)\nabla_y\xi-\eta(y)\nabla_x\xi\right\}, & \widehat{N}(x,y)=-2\left\{\eta(x)\nabla_y\xi+\eta(y)\nabla_x\xi\right\};\\
\mathcal{F}_9: & N(x,y)=2\left\{\eta(x)\nabla_y\xi-\eta(y)\nabla_x\xi\right\}, & \widehat{N}(x,y)=-2\left\{\eta(x)\nabla_y\xi+\eta(y)\nabla_x\xi\right\};\\
\mathcal{F}_{10}: & N(x,y)=-\eta(x)\varphi\left(\nabla_\xi\varphi\right)y+\eta(y)\varphi\left(\nabla_\xi\varphi\right)x, & \widehat{N}(x,y)=-\eta(x)\varphi\left(\nabla_\xi\varphi\right)y-\eta(y)\varphi\left(\nabla_\xi\varphi\right)x;\\
\mathcal{F}_{11}: & N(x,y)=\left\{\eta(x)\omega(\varphi y)-\eta(y)\omega(\varphi x)\right\}\xi, & \widehat{N}(x,y)=\left\{\eta(x)\omega(\varphi y)+\eta(y)\omega(\varphi x)\right\}\xi,
\end{array}
\tag{34}
$$

where $\theta^\sharp$ and $\omega^\sharp$ are the corresponding vectors of θ and ω with respect to g.

In [12], the tensor F is expressed by the Nijenhuis tensors on an arbitrary (M,φ,ξ,η,g) as follows:

$$
\begin{aligned}
F(x,y,z)=&-\frac{1}{4}\left\{N(\varphi x,y,z)+N(\varphi x,z,y)+\widehat{N}(\varphi x,y,z)+\widehat{N}(\varphi x,z,y)\right\}\\
&+\frac{1}{2}\eta(x)\left\{N(\xi,y,\varphi z)+\widehat{N}(\xi,y,\varphi z)+\eta(z)\widehat{N}(\xi,\xi,\varphi y)\right\}.
\end{aligned}
\tag{35}
$$

As corollaries, in the cases when $N=0$ or $\widehat{N}=0$, the latter relation takes the following form, respectively:

$$
\begin{aligned}
F(x,y,z)&=-\frac{1}{4}\left\{\widehat{N}(\varphi x,y,z)+\widehat{N}(\varphi x,z,y)\right\}+\frac{1}{2}\eta(x)\left\{\widehat{N}(\xi,y,\varphi z)+\eta(z)\widehat{N}(\xi,\xi,\varphi y)\right\},\\
F(x,y,z)&=-\frac{1}{4}\left\{N(\varphi x,y,z)+N(\varphi x,z,y)\right\}+\frac{1}{2}\eta(x)N(\xi,y,\varphi z).
\end{aligned}
$$

3.2. Natural connections on an almost contact B-metric manifold

Let D be a linear connection on (M,φ,ξ,η,g) and let us denote its torsion and potential (with respect to ∇) by T and Q, respectively. The corresponding tensors of type (0,3) are determined by $T(x,y,z)=g(T(x,y),z)$ and $Q(x,y,z)=g\left(Q(x,y),z\right)$. The relations (1) and (2) are valid.

In [24], it is given a classification of all linear connections on the almost contact B-metric manifolds with respect to their torsions T in 15 basic classes $\mathcal{T}_i$ $(i=1,\dots,15)$ (which are invariant and orthogonal subspaces with respect to the structure group) as follows:

$$
\begin{array}{rl}
\mathcal{T}_{1/2}: & T(\xi,y,z)=T(x,y,\xi)=0,\quad T(x,y,z)=-T(\varphi x,\varphi y,z)=-T(x,\varphi y,\varphi z),\quad t \overset{\neq}{=} 0;\\
\mathcal{T}_{3}: & T(\xi,y,z)=T(x,y,\xi)=0,\quad T(x,y,z)=-T(\varphi x,\varphi y,z)=T(x,\varphi y,\varphi z);\\
\mathcal{T}_{4/5}: & T(\xi,y,z)=T(x,y,\xi)=0,\quad T(x,y,z)-T(\varphi x,\varphi y,z)=\underset{x,y,z}{\mathfrak{S}} T(x,y,z)=0,\quad t \overset{\neq}{=} 0;\\
\mathcal{T}_{6}: & T(\xi,y,z)=T(x,y,\xi)=0,\quad T(x,y,z)-T(\varphi x,\varphi y,z)=\underset{x,y,z}{\mathfrak{S}} T(\varphi x,y,z)=0;\\
\mathcal{T}_{7/8}: & T(x,y,z)=\eta(z)T(\varphi^2x,\varphi^2y,\xi),\quad T(x,y,\xi)=\mp T(\varphi x,\varphi y,\xi);\\
\mathcal{T}_{9/10}: & T(x,y,z)=\eta(x)T(\xi,\varphi^2y,\varphi^2z)-\eta(y)T(\xi,\varphi^2x,\varphi^2z),\\
 & T(\xi,y,z)=T(\xi,z,y)=-T(\xi,\varphi y,\varphi z),\quad t \overset{\neq}{=} 0,\quad t^* \underset{\neq}{=} 0;\\
\mathcal{T}_{11}: & T(x,y,z)=\eta(x)T(\xi,\varphi^2y,\varphi^2z)-\eta(y)T(\xi,\varphi^2x,\varphi^2z),\\
 & T(\xi,y,z)=T(\xi,z,y)=-T(\xi,\varphi y,\varphi z),\quad t=0,\quad t^*=0;\\
\mathcal{T}_{12}: & T(x,y,z)=\eta(x)T(\xi,\varphi^2y,\varphi^2z)-\eta(y)T(\xi,\varphi^2x,\varphi^2z),\quad T(\xi,y,z)=-T(\xi,z,y)=-T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{13/14}: & T(x,y,z)=\eta(x)T(\xi,\varphi^2y,\varphi^2z)-\eta(y)T(\xi,\varphi^2x,\varphi^2z),\quad T(\xi,y,z)=\pm T(\xi,z,y)=T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{15}: & T(x,y,z)=\eta(z)\left\{\eta(y)\hat{t}(x)-\eta(x)\hat{t}(y)\right\},
\end{array}
$$

where the torsion forms associated with T are defined by

$$t(x)=g^{ij}T(x,e_i,e_j),\quad t^*(x)=g^{ij}T(x,e_i,\varphi e_j),\quad \hat{t}(x)=T(x,\xi,\xi).$$

Moreover, in [24] there are explicitly given the components T_i of $T\in\mathcal{T}$ in $\mathcal{T}_i$ ($i=1,\dots,15$).

A linear connection D is called a *natural connection* on (M,φ,ξ,η,g) if the almost contact structure and the B-metric are parallel with respect to D, i.e. $D\varphi=D\xi=D\eta=Dg=0$ [19]. As a corollary, we have also $D\widetilde{g}=0$. According to [24], a necessary and sufficient condition for a linear connection D to be natural on (M,φ,ξ,η,g) is $D\varphi=Dg=0$.

It is easy to establish (see, e.g. [19]) that a linear connection D is a natural connection on an almost contact B-metric manifold if and only if

$$Q(x,y,\varphi z)-Q(x,\varphi y,z)=F(x,y,z),\quad Q(x,y,z)=-Q(x,z,y).$$

Let us remark that the condition a linear connection to be natural does not imply that some of the basic classes $\mathcal{T}_i$ ($i=1,\dots,15$) to be empty for natural connections.

In [24], it is proved that an almost contact B-metric manifold $M=(M,\varphi,\xi,\eta,g)\in\mathcal{F}_i\setminus\mathcal{F}_0$ is normal, i.e. $N=0$, (respectively, has $\widehat{N}=0$) if the torsion of an arbitrary natural connection on M belongs to $\mathcal{T}_4\oplus\mathcal{T}_5\oplus\mathcal{T}_9\oplus\mathcal{T}_{10}\oplus\mathcal{T}_{11}$ (respectively, $\mathcal{T}_3\oplus\mathcal{T}_7$).

3.3. The φB-connection and the φ-canonical connection

In [21], it is introduced a natural connection on (M,φ,ξ,η,g) by

$$\dot{D}_xy=\nabla_xy+\dot{Q}(x,y),\quad \dot{Q}(x,y)=\frac{1}{2}\Big\{(\nabla_x\varphi)\,\varphi y+(\nabla_x\eta)\,y\,\xi\Big\}-\eta(y)\nabla_x\xi. \tag{36}$$

In [22], the connection determined by (36) is called a *φB-connection*. It is studied for some classes of the considered manifolds in [16, 17, 20–22] with respect to properties of the torsion and the curvature as well as the conformal geometry. The restriction of the φB-connection $\dot{D}$ on H coincides with the B-connection $\dot{\nabla}'$ on the corresponding almost complex Norden manifold, given in (19) and studied for the class $\mathcal{W}_1$ in [6].

The torsion of the φB-connection has the form

$$\dot{T}(x,y,z)=\frac{1}{2}\Big\{F(x,\varphi y,z)+\eta(z)F(x,\varphi y,\xi)+2\eta(x)F(y,\varphi z,\xi)\Big\}_{[x\leftrightarrow y]}. \tag{37}$$

Then it belongs to $\mathcal{T}_3 \oplus \mathcal{T}_4 \oplus \cdots \oplus \mathcal{T}_{15}$, according to [24].

Using (35), (37) and the orthonormal decomposition $x = hx + vx$, where $hx = -\varphi^2 x$, $vx = \eta(x)\xi$, we give the expression of the torsion of the φB-connection in terms of the Nijenhuis tensors as follows

$$\begin{aligned}\dot{T}(x,y,z) = \frac{1}{8}\Big\{&N(hx,hy,hz) + \mathop{\mathfrak{S}}_{x,y,z} N(hx,hy,hz) + \widehat{N}(hz,hy,hx) - \widehat{N}(hz,hx,hy)\Big\}\\ + \frac{1}{4}\Big\{&2N(vx,hy,hz) + N(hy,hz,vx) + 2\widehat{N}(vx,hy,hz) + N(hy,hz,vx)\\ &+ N(hx,hy,vz) + N(vz,hx,hy) - \widehat{N}(vz,hx,hy) - 2\widehat{N}(vz,vx,hy)\Big\}_{[x\leftrightarrow y]}.\end{aligned} \tag{38}$$

Taking into account (37), (38) and (34), we obtain for the manifolds from $\mathcal{F}_3 \oplus \mathcal{F}_7$ the following

$$\dot{T}(x,y,z) = \frac{1}{8}\Big\{N(hx,hy,hz) + \mathop{\mathfrak{S}}_{x,y,z} N(hx,hy,hz)\Big\} + \frac{1}{4}\Big\{N(hx,hy,vz) + \mathop{\mathfrak{S}}_{x,y,z} N(hx,hy,vz)\Big\}. \tag{39}$$

Therefore, using the notation $hN(x,y,z) = N(hx,hy,hz)$, for the basic classes with vanishing $\widehat{N}$ we have:

$$\mathcal{F}_3: \quad \dot{T} = \frac{1}{8}\Big\{hN + \mathfrak{S}hN\Big\}, \qquad \mathcal{F}_7: \quad \dot{T} = \frac{1}{2}\Big\{\mathrm{d}\eta \otimes \eta + \eta \wedge \mathrm{d}\eta\Big\}. \tag{40}$$

A natural connection $\ddot{D}$ is called a *φ-canonical connection* on (M,φ,ξ,η,g) if its torsion $\ddot{T}$ satisfies the following identity: [23]

$$\begin{aligned}\Big\{\ddot{T}(x,y,z) - \ddot{T}(x,\varphi y,\varphi z) - \eta(x)\Big\{\ddot{T}(\xi,y,z) - \ddot{T}(\xi,\varphi y,\varphi z)\Big\}&\\ - \eta(y)\Big\{\ddot{T}(x,\xi,z) - \ddot{T}(x,z,\xi) - \eta(x)\ddot{T}(z,\xi,\xi)\Big\}\Big\}_{[y\leftrightarrow z]} &= 0.\end{aligned}$$

Let us remark that the restriction of the φ-canonical connection $\ddot{D}$ on the contact distribution H is the unique canonical connection $\ddot{\nabla}'$ with torsion given in (22) on the corresponding almost complex Norden manifold studied in [8].

In [23], it is constructed a linear connection $\ddot{D}$ as follows:

$$g(\ddot{D}_x y,z) = g(\nabla_x y,z) + \ddot{Q}(x,y,z), \quad \ddot{Q}(x,y,z) = \dot{Q}(x,y,z) - \frac{1}{8}\left\{N(\varphi^2 z,\varphi^2 y,\varphi^2 x) + 2N(\varphi z,\varphi y,\xi)\eta(x)\right\}.$$

It is a natural connection on (M,φ,ξ,η,g) and its torsion is

$$\ddot{T}(x,y,z) = \dot{T}(x,y,z) + \frac{1}{8}\left\{N(hz,hy,hx) + 2N(hz,hy,vx)\right\}_{[x\leftrightarrow y]},$$

which is equivalent to

$$\ddot{T}(x,y,z) = \dot{T}(x,y,z) + \frac{1}{8}\Big\{N(hx,hy,hz) - \mathop{\mathfrak{S}}_{x,y,z} N(hx,hy,hz)\Big\} + \frac{1}{4}\Big\{N(hx,hy,vz) - \mathop{\mathfrak{S}}_{x,y,z} N(hx,hy,vz)\Big\}. \tag{41}$$

Obviously, $\ddot{D}$ is a φ-canonical connection on (M,φ,ξ,η,g) and it is unique. Moreover, the torsion forms of the φ-canonical connection coincide with those of the φB-connection.

In [23], it is proved that the φ-canonical connection and the φB-connection coincide on an almost contact B-metric manifold if and only if $N(hx,hy)$ vanishes, or equivalently, (M,φ,ξ,η,g) belongs to the class $\mathcal{U}_0 = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$. In other words, bearing in mind (38), the torsions of the φ-canonical connection and the φB-connection on a manifold from $\mathcal{U}_0$ have the form

$$\begin{aligned}\ddot{T}(x,y,z) = \dot{T}(x,y,z) = \frac{1}{8}\Big\{&\widehat{N}(hz,hy,hx) - \widehat{N}(hz,hx,hy)\Big\} + \frac{1}{4}\Big\{2N(vx,hy,hz) + N(vz,hx,hy)\\ &+ 2\widehat{N}(vx,hy,hz) - \widehat{N}(vz,hx,hy) - 2\widehat{N}(vz,vx,hy)\Big\}_{[x\leftrightarrow y]}.\end{aligned}$$

The torsions $\dot{T}$ and $\ddot{T}$ are different each other on a manifold belonging to the only basic classes $\mathcal{F}_3$ and $\mathcal{F}_7$ as well as to their direct sums with other classes. For $\mathcal{F}_3 \oplus \mathcal{F}_7$, using (39) and (41), we obtain the form of the torsion of the φ-canonical connection as follows

$$\ddot{T}(x, y, z) = \frac{1}{4}N(hx, hy, hz) + \frac{1}{2}N(hx, hy, vz).$$

Therefore, using (34), the torsion of the φ-canonical connection for $\mathcal{F}_3$ and $\mathcal{F}_7$ is expressed by

$$\mathcal{F}_3 : \quad \ddot{T} = \frac{1}{4}hN, \qquad \mathcal{F}_7 : \quad \ddot{T} = \mathrm{d}\eta \otimes \eta. \tag{42}$$

The general contactly conformal transformations of an almost contact B-metric structure are defined by

$$\bar{\xi} = e^{-w}\xi, \quad \bar{\eta} = e^{w}\eta, \quad \bar{g}(x, y) = e^{2u}\cos 2v\, g(x, y) + e^{2u}\sin 2v\, g(x, \varphi y) + (e^{2w} - e^{2u}\cos 2v)\eta(x)\eta(y), \tag{43}$$

where u, v, w are differentiable functions on M [17]. These transformations form a group denoted by G. If $w = 0$, we obtain the contactly conformal transformations of the B-metric, introduced in [20]. By $v = w = 0$, the transformations (43) are reduced to the usual conformal transformations of g.

Let us remark that G can be considered as a contact complex conformal gauge group, i.e. the composition of an almost contact group preserving H and a complex conformal transformation of the complex Riemannian metric $\overline{g^{\mathbb{C}}} = e^{2(u+iv)}g^{\mathbb{C}}$ on H.

Note that the normality condition $N = 0$ is not preserved by G. In [23], it is established that the tensor $N(\varphi\cdot, \varphi\cdot)$ is an invariant of G on any almost contact B-metric manifold and $\mathcal{U}_0$ is closed with respect to G. By direct computations is established there that each of $\mathcal{F}_i$ $(i = 1, 2, \ldots, 11)$ is closed by the action of the subgroup G_0 of G defined by the conditions $\mathrm{d}u \circ \varphi^2 + \mathrm{d}v \circ \varphi = \mathrm{d}u \circ \varphi - \mathrm{d}v \circ \varphi^2 y = \mathrm{d}u(\xi) = \mathrm{d}v(\xi) = \mathrm{d}w \circ \varphi = 0$. Moreover, G_0 is the largest subgroup of G preserving θ, θ^*, ω and $\mathcal{F}_0$. Moreover, the torsion of the φ-canonical connection is invariant with respect to the general contactly conformal transformations if and only if these transformations belong to G_0 [23].

Bearing in mind the invariance of $\mathcal{F}_i$ $(i = 1, 2, \ldots, 11)$ and $\ddot{T}$ with respect to the transformations of G_0, each of the basic classes of $(M, \varphi, \xi, \eta, g)$ is characterized by the torsion of the φ-canonical connection as follows: [23]

$$\begin{array}{rl}
\mathcal{F}_1 : & \ddot{T}(x, y) = \frac{1}{2n}\left\{\ddot{t}(\varphi^2 x)\varphi^2 y - \ddot{t}(\varphi^2 y)\varphi^2 x + \ddot{t}(\varphi x)\varphi y - \ddot{t}(\varphi y)\varphi x\right\}; \\
\mathcal{F}_2 : & \ddot{T}(\xi, y) = 0, \quad \eta\left(\ddot{T}(x, y)\right) = 0, \quad \ddot{T}(x, y) = \ddot{T}(\varphi x, \varphi y), \quad \ddot{t} = 0; \\
\mathcal{F}_3 : & \ddot{T}(\xi, y) = 0, \quad \eta\left(\ddot{T}(x, y)\right) = 0, \quad \ddot{T}(x, y) = \varphi\ddot{T}(x, \varphi y); \\
\mathcal{F}_4 : & \ddot{T}(x, y) = \frac{1}{2n}t'^{*}(\xi)\left\{\eta(y)\varphi x - \eta(x)\varphi y\right\}; \\
\mathcal{F}_5 : & \ddot{T}(x, y) = \frac{1}{2n}t'(\xi)\left\{\eta(y)\varphi^2 x - \eta(x)\varphi^2 y\right\}; \\
\mathcal{F}_6 : & \ddot{T}(x, y) = \eta(x)\ddot{T}(\xi, y) - \eta(y)\ddot{T}(\xi, x), \quad \ddot{T}(\xi, y, z) = \ddot{T}(\xi, z, y) = -\ddot{T}(\xi, \varphi y, \varphi z); \\
\mathcal{F}_{7/8} : & \ddot{T}(x, y) = \eta(x)\ddot{T}(\xi, y) - \eta(y)\ddot{T}(\xi, x) + \eta(\ddot{T}(x, y))\xi, \\
& \ddot{T}(\xi, y, z) = -\ddot{T}(\xi, z, y) = \mp\ddot{T}(\xi, \varphi y, \varphi z) = \frac{1}{2}\ddot{T}(y, z, \xi) = \mp\frac{1}{2}\ddot{T}(\varphi y, \varphi z, \xi); \\
\mathcal{F}_{9/10} : & \ddot{T}(x, y) = \eta(x)\ddot{T}(\xi, y) - \eta(y)\ddot{T}(\xi, x), \quad \ddot{T}(\xi, y, z) = \pm\ddot{T}(\xi, z, y) = \ddot{T}(\xi, \varphi y, \varphi z); \\
\mathcal{F}_{11} : & \ddot{T}(x, y) = \left\{\hat{\ddot{t}}(x)\eta(y) - \hat{\ddot{t}}(y)\eta(x)\right\}\xi.
\end{array}$$

According to the classification of the torsions in [24] and the characterization above, we have that the correspondence between the classes $\mathcal{F}_i$ of M and the classes $\mathcal{T}_j$ of the torsion $\ddot{T}$ of the φ-canonical connection on $M = (M, \varphi, \xi, \eta, g)$ is given as follows: [23]

$$\begin{array}{lll}
M \in \mathcal{F}_0 \Leftrightarrow \ddot{T} \in \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_6 \oplus \mathcal{T}_{12}; & M \in \mathcal{F}_4 \Leftrightarrow \ddot{T} \in \mathcal{T}_{10}; & M \in \mathcal{F}_8 \Leftrightarrow \ddot{T} \in \mathcal{T}_8 \oplus \mathcal{T}_{14}; \\
M \in \mathcal{F}_1 \Leftrightarrow \ddot{T} \in \mathcal{T}_4; & M \in \mathcal{F}_5 \Leftrightarrow \ddot{T} \in \mathcal{T}_9; & M \in \mathcal{F}_9 \Leftrightarrow \ddot{T} \in \mathcal{T}_{13}; \\
M \in \mathcal{F}_2 \Leftrightarrow \ddot{T} \in \mathcal{T}_5; & M \in \mathcal{F}_6 \Leftrightarrow \ddot{T} \in \mathcal{T}_{11}; & M \in \mathcal{F}_{10} \Leftrightarrow \ddot{T} \in \mathcal{T}_{14}; \\
M \in \mathcal{F}_3 \Leftrightarrow \ddot{T} \in \mathcal{T}_3; & M \in \mathcal{F}_7 \Leftrightarrow \ddot{T} \in \mathcal{T}_7 \oplus \mathcal{T}_{12}; & M \in \mathcal{F}_{11} \Leftrightarrow \ddot{T} \in \mathcal{T}_{15}.
\end{array}$$

3.4. The φKT-connection

In [19], it is introduced a natural connection $\ddot{D}$ on (M,φ,ξ,η,g), called a *φKT-connection*, which torsion $\ddot{T}$ is totally skew-symmetric, i.e. a 3-form. There it is proved that the φKT-connection exists on an almost contact B-metric manifold if and only if $\widehat{N}$ vanishes on it, i.e. when $(M,\varphi,\xi,\eta,g)\in\mathcal{F}_3\oplus\mathcal{F}_7$. The φKT-connection is the odd-dimensional analogue of the KT-connection introduced in [27] on the corresponding class of quasi-Kähler manifolds with Norden metric. The unique φKT-connection $\ddot{D}$ is determined by

$$g(\ddot{D}_x y,z)=g(\nabla_x y,z)+\frac{1}{2}\ddot{T}(x,y,z),$$

where the torsion is defined by

$$\ddot{T}(x,y,z)=-\frac{1}{2}\underset{x,y,z}{\mathfrak{S}}\Big\{F(x,y,\varphi z)-3\eta(x)F(y,\varphi z,\xi)\Big\}=\frac{1}{4}\underset{x,y,z}{\mathfrak{S}}N(x,y,z)+\frac{1}{2}\left(\eta\wedge\mathrm{d}\eta\right)(x,y,z). \tag{44}$$

Obviously, the torsion forms of the φKT-connection are zero.

The torsion $\ddot{T}$ of the φKT-connection belongs to $\mathcal{T}_3\oplus\mathcal{T}_6\oplus\mathcal{T}_7\oplus\mathcal{T}_{12}$, according to [24].

From (44) and (34), for the classes $\mathcal{F}_3$ and $\mathcal{F}_7$ we obtain

$$\mathcal{F}_3:\quad \ddot{T}=\frac{1}{4}\underset{x,y,z}{\mathfrak{S}}hN,\qquad \mathcal{F}_7:\quad \ddot{T}=\eta\wedge\mathrm{d}\eta. \tag{45}$$

As mentioned above, the φB-connection and the φ-canonical connection coincide (i.e. $\dot{D}\equiv\ddot{D}$) if and only if (M,φ,ξ,η,g) belongs to $\mathcal{F}_i$, $i\in\{1,2,\dots,11\}\setminus\{3,7\}$ (where the φKT-connection $\ddot{D}$ does not exist).

For the rest basic classes $\mathcal{F}_3$ and $\mathcal{F}_7$ (where the φKT-connection exists), according to [24], it is valid that the φB-connection is the *average* connection of the φ-canonical connection and the φKT-connection, i.e. $\dot{D}=\frac{1}{2}\left\{\ddot{D}+\ddot{D}\right\}$. This relation holds also because of (40), (42) and (45).

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